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A N
E S S A Y
O N T H E
Resolution of Plain Triangles,
B Y
COMMON ARITHMETIC:
W I T H
A NEW AND CONCISE TABLE,
ADAPTED TO THE PURPOSE.

By HUGH WORTHINGTON, Jun.

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THOUGH the invention of *Logarithms* was one of the most useful discoveries in Mathematics, yet cases frequently occur in practice, wherein it is expedient to perform trigonometrical operations without the logarithmic canon. On this account, repeated attempts have been made, to substitute certain rules and proportions in *common arithmetic*, in the place of the natural and artificial sines and tangents. What these rules are, and on what principles they depend, is briefly shewn in the introductory parts of the following Essay; in which the author has endeavoured to collect the observations of preceding writers, and has added some remarks of his own. Hence it will be easy to extend the principles of analysis, and by comparing the several equations to construct a variety of *formulae*, as occasion shall require.* — Still, however, some farther improvement seemed necessary; the consideration of which gradually led the author to the plan proposed in the third section, which will be found on trial much more *easy* and *accurate*, than the methods

* Examples are given in the Appendix.

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methods formerly used. If it should be objected to this plan, that the use of *a table* is inconsistent with the nature of the arithmetical scheme, to this it may be replied—that as few persons have memory sufficient to retain the common rules without *transcribing* them, so the transcript of those rules would occupy as much *space* on paper as the annexed table; and therefore the one is as portable and convenient as the other.

The reader must not expect to find here *a system* of trigonometry, since that was foreign from the design of these pages, and would be totally superfluous, after the many excellent treatises upon this subject, already in print. If he is not accustomed to analytical investigations, he may omit a part, or the whole, of the *second* section: but it is taken for granted, that he has a general acquaintance with the common branches of mathematics.

ERRATUM.

P. 8, l. 21, —for E read C.

A N
'E S S A Y, &c.

S E C T. I.

THAT the reader may judge properly of the design of this Essay, it seems necessary to lay before him the rules *already* established by mathematicians, for the resolution of plain triangles, without having recourse to instruments, or voluminous tables.

The doctrine of *right-angled* triangles is generally distributed into *seven* cases; but these are reducible to *four*, when the operation does not depend upon the tables. I shall place them in the order most convenient to my purpose.

The FIRST CASE is that of *the legs being given to find the hypotenuse*: which is solved by *Euclid* 1 and 47; or “the square-root of the sum of the squares of the legs = hypotenuse.”

B

The

The SECOND CASE is, *the hypotenusè and one leg given, to find the other leg.* Here it is evident, that “the square-root of the difference of the squares of the hypotenusè and given leg must be the precise value of the other.” Or more easily thus; since the rectangle of the sum and difference of any two numbers is equal to the difference of their squares, “multiply the sum of the hypotenusè and one leg by their difference, —the square root of that product will be the other leg required.”

The THIRD CASE is, *the sides given, to find the angles:* and the rule is, as follows. “Half the longer of the two legs added to the hypotenusè, is always in proportion to 86, as the shorter leg is to its opposite angle.”

The FOURTH and LAST CASE is, *the angles and a side given, to find another side.* For the solution of which problem, the following rule has been proposed by some writers;*

* Particularly by Mr. *Emerson*, in his *Elements of Trigonometry*.

“ In any right angled triangle, as CAB (*Fig. 1.*) let a be the lesser angle in degrees; then as $\frac{57.3}{a} + \frac{3a}{1000} : 1 :: \text{hyp. AC} : \text{opposite side AB, nearly;}$ and as $1 : 1 - \frac{1}{2} \frac{aa}{10000} :: \text{hyp. AC} : \text{adjacent side BC, nearly.}$ ”

Mr. *Turner*, in his “ Plain Trigonometry rendered easy and familiar,” has revived the solution first given by Mr. *Henry Wilson*, in his *Navigation new modelled* (published about 1715): which is to this effect.—“ Divide 4 times the square of the complement of the angle, whose opposite side is either given or sought, by 300 added to 3 times the said complement; this quotient, added to the said angle, will give you an artificial number, called sometimes the *natural radius*, which will ever bear the same proportion to the hypotenuse, as that angle bears to its opposite side.—In angles under 45 degrees, the *artificial number* may be found easier thus: divide 3 times the square of the angle itself, whose opposite side is given or sought, by 1000; the quotient added to 57.3 (the *radius* of a circle, whose periphery is 360) that sum will be the artificial number required.”

It may not be improper to observe, that the latter part of Mr. Wilson's rule is *virtually* the same with the first proportion,

$$\frac{57.3}{a} + \frac{3a}{1000} : 1 :: \text{hypotenuse} : \text{opposite side.}$$

For since the equimultiples of quantities are to each other, as the quantities themselves, if the two first terms be multiplied by a , the analogy will stand thus; $57.3 + \frac{3aa}{1000} : a :: \text{hyp.} : \text{opposite side.}$

These are the chief rules hitherto given for the 4 cases of rectangular trigonometry.

Oblique triangles are treated after the same manner, by first dividing them into 2 right-angled triangles, by means of a perpendicular, which must always fall from the end of a given side, and opposite to a given angle. Here, let it be observed, (1.) The perpendicular will sometimes fall within, and sometimes without the triangle: when it falls within, it falls upon some part of the base, or longest side, as in *fig. 2*; but when it falls without, it falls upon one of the shorter sides continued, as in *fig. 3*. In either case, there are two right-angled triangles made, whose sides or angles are to be found

as

as above. (2.) The *segments* of the base, occasioned by the perpendicular falling upon it, are easily calculated by the following proportion; as the base is to the sum of the other two sides, so is the difference of the sides to the difference of the segments of the base: the half of this difference added to half the base gives the *greater* segment, and subtracted from it, the *less*.

S E C T. II.

HAVING enumerated in the former section the principal rules for the resolution of plain triangles, without recurring to tables of sines, tangents, or secants, I come now to shew the *grounds* on which they are built.

Among these rules, we need not take notice of the *first* and *second*, as every learner must be acquainted with the 47th proposition of Euclid's Elements. Therefore I begin with the *third*, in which it is asserted, that "the sum of the hypotenuse and half the

the longer leg is in the same proportion to 86, as the shorter leg is to its opposite angle." This rule was originally given by *Snellius*, who deduced it from the supposition of the chord and tangent of an arc (under 45°) being nearly equal to one another: a supposition, which can be true only in *very small arcs*, for in the case of 40° , making radius *unity*, the chord is but .68404, while the tangent is .83910. However this circumstance should not prejudice the reader against the rule itself, since it is found by *experiment*, that the answer produced by this proportion is *exact* enough for all common purposes.*

Besides, as Mr. Wilson has stated the matter, in his *Trigonometry improved* (p. 18)

* It comes so near the truth, as *very frequently* to give the answer right in *minutes*; and I cannot but think it matter of admiration, that a rule so *simple* (by much the simplest of any discussed in these pages) should be capable of this accuracy.—To give but one example. Let the 3 sides of a right-angled triangle be 1277, 1124, and 606: (where only 2 sides are known, the 3d must be found by calculation); now $1277 + 562 = 1839$. And $1839 : 86 :: 606 : 28.33$ + that is $28^{\circ}, 20'$: which is the exact answer by the logarithms, and leaves the remaining angle $= 61^{\circ}, 40'$.

there

there is no occasion to argue from such erroneous premises. The demonstration he there gives is grounded on the supposition of equality (not between the *chord* and *tangent* of an arc, but) between the *arc* itself, and a line much *less* than the *tangent*.

I shall quote his own words. "In the triangle ABC (*Fig. 4*) continue the longer leg CB both ways, and with the extent AC, and one foot in the least angle, describe the circle FAD; make $FG = FC$, erect the perpendicular or tangent line DE, and draw GA continued, till it cut the said tangent line in E.*

"Now if we grant the arch AD equal to the right line DE, and the diameter of a

* The reader must carefully observe, that though DE is a *tangent line* to the circle at the point D, yet it is by no means *the tangent* of the arch AD: and to convince him, how nearly it approaches to an equality with AD, even when the arch is 40° , I will subjoin the calculation in numbers. Putting CA, or CD = *unity*, we have $AB = .64278$, and $CB = .76604$, by the table of natural sines. Hence GD is = 3, and $GB = 2.76604$; and since $GB : BA :: GD : DE$, we shall get $.69714$ for the true measure of DE, which differs very little from $.69813$ the real length of the arch AD.—This view of the subject ought to restore the credit of *Snellius's* rule, which has needlessly suffered, by the absurd principles on which he grounded it.

circle

circle to be to the circumference as 113 to 355, the proportion follows thus :

“ In the similar triangles GBA, and GDE, it is as GB to BA, so GD to DE, (*Euclid. Lib. VI. Prop. 4.*) and if DA be equal to DE, it is as GB to BA, so GD to DA ; and because we are endeavouring to measure the angle BCA by the arch AD, it must be supposed to be an arch of a circle, the whole of which contains 360 parts or degrees, and its half DAF 180 ; and by the above-mentioned proportion its diameter is $114\frac{42}{11}$, and the half of that, *viz.* $57\frac{21}{11}$ is *radius*, and three times $57\frac{21}{11}$ is $171\frac{63}{11}$. Now, because by construction GD is 3 times *radius*, *viz.* $171\frac{63}{11}$, which for evenness of numbers we will call 172, and GF is equal to FC, equal to CA the hypotenuse, and CB is the longer leg, it will be as GB (twice the hypotenuse, and once the longer leg) to GD (thrice *radius*, or) 172, so BA the shorter leg, to DE, which is granted equal to DA, which (DA) is the measure of the angle E required.

“ Therefore, taking half the two first terms, it will be as the hypotenuse added to half the longer leg is to 86, (the half of 172) so the shorter leg is to its opposite angle.”

I shall

I shall next shew, how the same rule or proportion may be derived, in a very different manner, from a *fluxionary equation*: which may be thus expressed in the form of a

L E M M A.

If the length of any arch in a circle be denoted by z , its radius by r , and its versine by v , then $\dot{z} = \frac{r \dot{v}}{\sqrt{2rv - v^2}}$; whose fluent, in an infinite series, is $z = \sqrt{2rv} + \frac{v}{3 \cdot 4r} A + \frac{3^2 v}{5 \cdot 8r} B + \frac{5^2 v}{7 \cdot 12r} C + \&c.$ where A, B, C are the foregoing terms. *

Hence we may prove the following

P R O P O S I T I O N.

Fig. 5. *In any right-angled triangle, CAB,*
As the sum of the hypotenuse and half the
longer leg, $AC + \frac{1}{2} CB$:

To 86 ::

So the shorter leg, AB :

To its opposite angle, C , nearly.

* The demonstration of the Lemma is omitted, because it occurs in most treatises on the Doctrine of Fluxions. See Hodgson, p. 228. Simson, p. 165, &c.

C

DEMON.

D E M O N.

From C describe the arch AD: then BD (or v) is the versè sine = CA—CB, and AC (or b) is the radius. By the *lemma* the arch

$$AD = \sqrt{2bv} + \frac{v}{3.4} A + \frac{9v}{5.8b} B, \&c. \quad \text{But as}$$

half the periphery ($3.14159b$): $180 :: AD$:

$$\text{degrees in } AD = \frac{180 AD}{3.14159b} = \frac{57.2957}{b} \times AD = \frac{57.29}{b}$$

$$\times : \sqrt{2bv} + \frac{v}{12b} A, \&c.$$

Let $CB=b$, $AB=c$, a = angle C in degrees. By nature of the circle, $c = \sqrt{2bv - vv}$

$$= (\text{by evolution}) \sqrt{2bv} - \frac{v}{4b} \sqrt{2bv}, \&c.$$

Now we have proved $a = \frac{57.3}{b} \times : \sqrt{2bv} +$

$\frac{v}{12b} \sqrt{2bv}$ nearly: multiply these two

equations alternately, the first by a , the se-

cond by c , then, $a \times : \sqrt{2bv} - \frac{v}{4b} \sqrt{2bv} =$

$\frac{57.3c}{b} \times : \sqrt{2bv} + \frac{v}{12b} \sqrt{2bv}$; divide by $\sqrt{2bv}$,

$a \times 1 - \frac{v}{4b} = \frac{57.3c}{b} \times 1 + \frac{v}{12b}$; again, $\frac{57.3c}{b} = a \times$

$$\frac{1 - \frac{v}{4b}}{1 + \frac{v}{12b}} = a \times 1 - \frac{v}{3b} \&c. = a \times \frac{3b-v}{3b} = a \times \frac{3b-b+v}{3b}$$

(for

(for $v = b - b$) $= a \times \frac{2b+b}{3b}$. Multiply by $3b$, then, $a \times \overline{2b+b} = 171.9c = 172c$ nearly; that is, $a \times b + \frac{b}{2} = 86c$, or $b + \frac{b}{2} : 86 :: c : a$. Q. E. D. *

It now remains, to investigate the rules laid down for the solution of the *fourth* case of right angled triangles; *viz. the angles and a side given to find another side*. And here again we must have recourse to fluxions, by means of which we derive another convenient and most useful

L E M M A.

If the length of any arch in a circle be denoted by z , its radius by r , and its right-line by y , then $\dot{z} = \frac{r\dot{y}}{\sqrt{rr-yy}}$. And the fluent being taken, $z = y + \frac{y^3}{2.3r^2} + \frac{3y^5}{2.4.5r^4} + \frac{3.5y^7}{2.4.6.7r^6} + \&c.$ Hence by reversion of series, $y = z - \frac{z^3}{2.3r^2} + \frac{z^5}{2.3.4.5r^4} - \frac{z^7}{2.3.4.5.6.7r^6} + \&c.$ And the cosine of the arch ($= \sqrt{rr-yy}$) $= r - \frac{z^2}{2r} + \frac{z^4}{2.3.4r^3} - \frac{z^6}{2.3...6r^5} + \frac{z^8}{2....8r^7} - \&c.$

* Vid. *Emerson's Trigonometry*, p. 97:

By the assistance of this last equation, we shall be able to demonstrate the following

PROPOSITION.

Fig. 5. In any right-angled triangle, if a be the lesser angle in degrees, then as $\frac{57.3}{a} + \frac{3a}{1000} : 1 :: \text{hyp. AC} : \text{opposite side AB, nearly; and as } 1 : 1 - \frac{1\frac{1}{2}aa}{10000} :: \text{hyp. AC} : \text{adjacent side BC, nearly.}$

DEMON: PART I.

Let radius CA or CD = b , arch AD = z . It appears from the former prop. that the degrees in AD = $\frac{57.295}{b} \times \text{AD}$, or $a = \frac{57.29}{b} \times z$. Therefore $z = \frac{ab}{57.29}$, and putting $m = \frac{a}{57.29}$, $z = mb$. By the lemma to this proposition, cosine CB = $b - \frac{z^2}{2b} + \frac{z^4}{2.3.4b^3}$, &c. Therefore $CB^2 = bb - z^2 + \frac{z^4}{3b^2}$ &c: therefore $AB^2 (=CA^2 - CB^2) = z^2 - \frac{z^4}{3b^2}$, and extracting the root $AB = z - \frac{z^3}{6b^2} = (\text{putting } mb = z) mb - \frac{m^3b}{6}$, = (putting $n = 57.3$, so that $\frac{a}{n} = m$) $b \times \frac{a}{n} - \frac{a^3}{6n^3}$. Divide by $\frac{a}{n} - \frac{a^3}{6n^3}$, then

$$\text{then } b = \frac{AB}{\frac{a}{n} - \frac{a^3}{6n^3}} = AB \times \frac{1}{\frac{a}{n} - \frac{a^3}{6n^3}} = A B$$

$$\times \frac{n}{a} + \frac{a}{6n} = AB \times : \frac{57.3}{a} + \frac{3a}{1031}. \text{ Or, } \frac{57.3}{a} + \frac{3a}{1000}$$

$$: 1 :: CA : AB, \text{ nearly. Q. E. D.}$$

P A R T II.

If $CB = b - \frac{2z}{2b}$ &c. and $z = mb$, then

$$CB = b - \frac{m^2 b^2}{2b} \&c. = CA \times \frac{1}{1 - \frac{m^2}{2}}, = CA$$

$$\times \frac{1}{1 - \frac{aa}{2nn}} = CA \times \frac{1}{1 - \frac{1.52aa}{10000}}; \text{ whence the}$$

proportion $1 : 1 - \frac{1.5aa}{10000} :: CA : BC.$

Q. E. D.

From the second part of this demonstration is derived what we generally call *Wilson's Rule*; (See the former §,) and the investigation is as follows. We have proved CB

$$= CA \times \frac{1}{1 - \frac{aa}{2nn}}; \text{ now put } c = 90^\circ, y = c - a,$$

and divide the foregoing equation by $1 - \frac{aa}{2nn}$. Then $CA = CB \times \frac{1}{1 + \frac{aa}{2nn}}$, and CA

$$\times y = CB \times y + \frac{4aa}{8} \cdot \frac{y}{nn}. \text{ But } \frac{8nn}{y} = \frac{8nn}{c-a} = \frac{8nn}{c}$$

+

+ $\frac{8n^2a}{cc}$ &c. = $292 + 3a$, which may be called $300 + 3a$ nearly. Therefore, $CA \times y = CB \times y + \frac{4aa}{300+3a}$. Which is the rule itself, expressed in algebraic terms.

There is another method (called *Förster's* rule) of determining the relation between the sides and angles of a right-angled triangle, which, though not so convenient as the above, yet appears deserving of a place here. Suppose $n=57.3$, b =hypothenufe, b =longer leg, and a =degrees in the smaller angle; and let the side opposite that angle be constantly put = *unity*. Then

$$b = \frac{2n}{a} - \sqrt{\frac{nn}{aa} - \frac{1}{3}};$$

$$\text{and } b = \frac{3n}{a} - 2b.$$

These two equations are derived from *Snellius's* proportion; thus ——— If $b + \frac{b}{2} : 1 :: \frac{3n}{2} : a$, then $\frac{3n}{2} = ba + \frac{ba}{2}$. Consequently $3n = 2ba + ba$, and $\frac{3n}{a} = 2b + b$. Hence $b = \frac{3n}{a} - 2b$, which is the *second formula*, so that if b be known, b may be deter-

determined. Again, since $2b = \frac{3n}{a} - b$, we have $b = \frac{3n}{2a} - \frac{b}{2} = \frac{3n}{2a} - \sqrt{\frac{bb-1}{2}}$, and therefore $\frac{3n}{a} - 2b = \sqrt{bb-1}$; this equation reduced gives $b = \frac{2n}{a} - \sqrt{\frac{nn}{aa} - \frac{1}{3}}$, which is the *first* of the above formulæ.*

* As in these equations, n and $\frac{1}{3}$ are known and constant quantities, their *numerical* value may as well be substituted. That is; $b = \frac{114.6}{a} - \sqrt{\frac{3283.29}{aa} - .3333}$ and $b = \frac{171.9}{a} - 2b$.

N. B. Since by this rule we only gain the *relative* value of b and b , it will be necessary to multiply them by the *true* value of the shorter leg in any given case, in order to obtain a proper solution of the problem.

SECT.

S E C T. III.

IT appears from the preceding pages, that one of the most essential problems, in the resolution of right-angled triangles, is that of *the angles and a side being given, to find another side*. This has been the subject of repeated disquisition; and different writers have offered different methods of solving it. Yet, after all, I am dissatisfied with the *long calculations* they frequently require, and still more so with occasional proofs of their *incorrectness*.*—To remove these inconveniences, I once thought of an *abridgement* of the *natural sines*, which would answer the purpose sufficiently, if they could be reduced to a very small scale; but in the higher parts of the quadrant, so many decimal places are requisite to distinguish the sines with accuracy, that I am persuaded the attempt will ever prove abortive. This circumstance, together with a close examination of the *proportional increase* of sines (considered in the

* For example; the natural radius of 43° , as found by *Wilson's rule*, is 62.84, whereas it ought to be 63.05.

light of fluents) induced me to construct a table of *radii*, instead of *sines*, for every degree in the quadrant. For I found, by numberless trials, that though the rate of increase was exceedingly irregular among the *sines*, yet on the supposition of every arch having a distinct radius, the fluxions of such radii were remarkably uniform. For example; if the sine of an arch (radius being 100,000) be considered as a *flowing* or increasing quantity, it changes so quick through the first *third* of its course, as to rise from 29 (sine of one minute) to 50,000 (sine of 30°). But if the natural radius be made a variable quantity, it only passes through the augmentation of 2.7, in the same portion of the quadrant: for at 1' it is 57.3, and at 30° it is no more than 60. I allow that when the arch becomes much greater, the fluxion of the radius is accelerated, but it never increases by one entire *unit* in any single degree. This being the fact, I was led to hope, that if a table of radii was calculated only for *degrees*, the value of any radius for the intervening *minutes* might be found by proportion. Experiment has confirmed this theory beyond a doubt: and I can put the reader into an easy

D

method

method of satisfying himself of its truth; for let him make trial of the *semi-degrees* only, and he will find that the radius taken by *proportion*, universally corresponds with that found by the *logarithms*: and therefore it may fairly be concluded, that all the minutes above and below the *middle point* will answer with exactness.

These things being premised, I proceed to explain more particularly the nature of the table subjoined to this section. It is divided, for the sake of convenience, into three columns, and each of those columns into three more. Of these, the *first* is marked A, as it contains the degrees in every arch or angle of the quadrant, in their natural series; the *second* is entitled R, containing the radii of those degrees, both in whole numbers and decimals, which have been calculated with the utmost care from the best edition of *Vlacq's Logarithms*; the *third* is marked D, because it expresses the difference between one radius and another, the use of which will appear quickly: and here let it be noted, that the difference of any two radii is placed opposite the *larger* of them. — At the

the back of the table, you have the decimal value of every minute in a degree, which though not essential to the table, will be found very convenient for practice, as it saves the trouble of reduction in working a problem. Here it is proper to inform the reader, that whenever the last decimal figure is either 3 or 6, the quotient would continue the same *ad infinitum*.

With respect to the *practical use* of the table, many words are not needful. When the given angle consists only of *degrees*, the natural radius is found by inspection: Thus the radius of 38° is 61.72. But when the angle includes *minutes* as well as degrees, then the following operation is necessary; look into the table of minutes for the decimal value of the number assigned; call it m ; then look into the column marked D, for the difference between the 2 nearest radii, and call that d ; $m \times d$ added to the smaller radius, will give the true radius required. As for instance, suppose the angle given to be $43^\circ, 25$; in this case $m = .416$, and $d = .29$, whose product is .12064, which being added to 63.05 (the radius of 43°) we

D 2
have

have 63.17 for the radius sought. Or suppose the given angle was $59^{\circ} 47'$; then $m = 783$, $d = .45$, and $md = .35235$: add this to 68.83 (radius of 59°) and you will have 69.18 for the term required.

The reason of multiplying m by d will be manifest, by stating at length the following proportion; as the whole number of minutes in a degree, to the whole difference of radii in the table, so is the fractional value of any assigned number of minutes, to the correspondent difference which is sought. Here the first term is expressed by *unity*; therefore, the fourth is simply the product of the second and third; on which account, the operation must always be easy.

I shall now give some examples of the application of this table to that case in right-angled triangles; “ the angles and hypothenuse being given, to find either leg;” *et vice versa*.

EXAM-

EXAMPLE I.

In the right-angled triangle ABC, (*Fig. 6*).

Given $\left\{ \begin{array}{l} A = 15^{\circ}, 07' \\ BC = 86 \end{array} \right\}$ Required AB.

$$.116 = m$$

$$.09 = d$$

$$.01044$$

$$57.95$$

$$57.96$$

Angle A : BC :: Radius

$$15.116 \text{ --- } 86 \text{ --- } 57.96$$

$$86$$

$$34776$$

$$46368$$

$$15.116) 4984.56 (329.7 = AB$$

$$44976$$

$$147440$$

$$113960$$

$$8148$$

EXAMPLE II. (*Fig. 6*.)

Given $\left\{ \begin{array}{l} A = 33^{\circ}, 45' \\ AB = 987 \end{array} \right\}$ Required BC.

$$.75 = m$$

$$\begin{array}{r}
 .75 = m \\
 .21 = d \\
 \hline
 75 \\
 150 \\
 \hline
 .1575 \\
 60.59 \\
 \hline
 60.74
 \end{array}$$

Radius : AB :: Angle A

$$\begin{array}{r}
 60.74 \text{ --- } 987 \text{ --- } 33.75 \\
 \phantom{60.74 \text{ --- } 987 \text{ --- }} 987 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 23625 \\
 27000 \\
 \hline
 30375
 \end{array}$$

$$60.74)33311.25(548 = BC.$$

$$\begin{array}{r}
 29412 \\
 \hline
 51165
 \end{array}$$

$$2573$$

EXAMPLE III. (Fig. 6.)

Given $\left\{ \begin{array}{l} B = 56^\circ, 20' \\ AB = 496 \end{array} \right\}$ Required AC.

$$*d = .42$$

$$\frac{1}{3}d = .14$$

$$\begin{array}{r}
 67.54 \\
 \hline
 67.68
 \end{array}$$

* When the *minutes* are an *aliquot part* of a degree (as a *half*, a *third*, a *fourth*, &c.) there is no occasion for a formal multiplication of *m* by *d*.

Radius

Radius : AB :: Angle B

$$67.68 \text{ --- } 496 \text{ --- } 56.33$$

496

33798

50697

22532

$$67.68) 27939.68 (412 = AC.$$

8676

19088

5552

EXAMPLE IV. (Fig. 6.)

Given $\left\{ \begin{array}{l} B = 74^\circ, 53' \\ AC = 315 \end{array} \right\}$ Required AB.

$$.883 = m$$

$$.66 = d$$

5298

5298

.58278

76.98

77.56

Angle

Angle B : AC :: Rad.

$$74.883 \text{ --- } 315 \text{ --- } 77.56$$

315

38780

7756

23268

$$74.883)24431.40(326=AB.$$

196650

468840

19542

Having so far illustrated the manner of working by the table, it is proper to shew the difference in point of *labour*, between the old method and the new: To this end I shall resolve the last of the above cases by Mr. *Wilson's* rule.

The complement of the given angle is 15°, 07', or in decimals 15.116.

15.116

3

45.348

300

345.348

15.116

[25]

$$\begin{array}{r}
 15.116 \\
 15.116 \\
 \hline
 90696 \\
 15116 \\
 15116 \\
 75580 \\
 15116 \\
 \hline
 228.493456 \\
 4
 \end{array}$$

$$\begin{array}{r}
 345.34)913.973824(2.646+ \\
 223293 \\
 160898 \\
 227622 \\
 \hline
 20418
 \end{array}$$

$$\begin{array}{r}
 2.646 \\
 74.883 \\
 \hline
 \end{array}$$

77.529=Radius, or more conveniently 77.53

Ang. B : AC :: Rad.

$$\begin{array}{r}
 74.883 \text{ --- } 315 \text{ --- } 77.53 \\
 315 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 38765 \\
 7753 \\
 23259 \\
 \hline
 \end{array}$$

$$74.883)24421.95(326 \text{ AB.}$$

$$\begin{array}{r}
 195705 \\
 459290 \\
 \hline
 \end{array}$$

9992

E

Thus

Thus it appears, that in a *very simple operation*, a great deal of trouble is saved, especially in the practice of division: to say nothing of superior accuracy; for had each operation been continued to decimal places in the quotient, the latter would have been found defective. But I shall advance a step farther, and shew the usefulness of the new table in a more *complicated case* of trigonometry (as performed by common arithmetic) and yet a case that frequently occurs; namely, WHEN THE ANGLES AND ONE LEG ARE GIVEN TO FIND THE OTHER LEG. According to the usual process, this cannot be solved by any direct rule in the first section, but must require a successive application of rules: as *first*, for finding the natural radius: *secondly* for finding the hypotenuse; and *thirdly* for finding the other leg by extraction of roots, which is to many persons a laborious task.

Now I shall point out a method, in which there is no occasion to find the hypotenuse, and wherein the simple *rule of three* is a substitute for evolution: and after giving two examples of the practice, in the case proposed,
I shall

I shall next apply it to the solution of a problem in *oblique trigonometry*; which will be allowed perhaps to be the strongest proof of the advantages resulting from the new table.

EXAMPLE I.

In the right-angled triangle ABC, (*Fig. 6*) we have given the angle $A = 37^{\circ}, 30'$, $B = 52^{\circ}, 30'$, and the leg $BC = 85$. Required the leg AC.

Here I look into the table for the *radii* of the two angles, and place them in order thus ;

$$\begin{array}{rcl} 37.5 & \text{---} & 61.6 \\ 52.5 & \text{---} & 66.17 \end{array}$$

As these numbers stand, the angles are considered as *sines*, and therefore each has a *distinct radius* : but it will be easy to throw out one of the radii, and reduce either of the sines to the correspondent value of the other sine. In general it is most convenient to reduce the *smaller term* to a conformity with the larger.

Therefore, in the present case,

$$61.6 : 37.5 :: 66.17 : 40.28, \text{ which may be called the } \textit{artificial} \text{ sine of } 37.5, \text{ with rad. } 66.17$$

$$\begin{array}{r} 37.5 \\ \hline 33085 \\ 46319 \\ 19851 \\ \hline \end{array}$$

$$\begin{array}{r} 61.6)2481.375 \\ 1737 \\ 5055 \\ \hline 127 \end{array}$$

Having found the *artificial sine*, I proceed to work on this common principle, "that the sides of a triangle are proportional to the sines of their opposite angles;" that is,

$$40.28 : 85 :: 52.5$$

$$\begin{array}{r} 85 \\ \hline 2625 \\ 4200 \\ \hline \end{array}$$

$$40.28)4462.5(110.7 = \text{AC, the leg required.}$$

$$\begin{array}{r} 4345 \\ 31700 \\ \hline 3504 \end{array}$$

The above question is taken from Mr. *Turner's Trigonometry* (p. 15); and to satisfy

tisfy the reader which of the two methods is the *shortest* and *easiest*, I shall transcribe the solution there given.

(1st.) Find the natural radius to $37^{\circ} 30'$.

$$\begin{array}{r}
 37.5 \\
 37.5 \\
 \hline
 1875 \\
 2625 \\
 1125 \\
 \hline
 1406.25 \\
 3 \\
 \hline
 1000) 4.218.75 \\
 57.3 \\
 \hline
 61.51875
 \end{array}$$

But 61.52 is exact enough in practice.

(2^{dly}.) Find the hypotenuse AB.

Ang. A : BC :: Radius

$$\begin{array}{r}
 37.5 \text{ --- } 85 \text{ --- } 61.52 \\
 \phantom{37.5 \text{ --- } 85 \text{ --- }} 85 \\
 \hline
 \phantom{37.5 \text{ --- } 85 \text{ --- }} 30760 \\
 49216 \\
 \hline
 5229.20
 \end{array}$$

37.5

[30]

$$37.5)5229.20(139.44 = AB$$

1479

3542

1670

1700

200

(3dly.) Find the side AC.

To hyp. 139.44

Add perp. 85

Sum 224.44

Multiply by diff. 54.44

89776

89776

89776

112220

Extract the root $\dot{1}2\dot{2}1\dot{8}.5\dot{1}3\dot{6}(110.537=AC.$

1.....

21) 22

21

2205) 11851

11025

22103) 82636

66309

221067) 1632700

1547469

85231

EX-

EXAMPLE II.

In the right-angled triangle ABC, (*Fig. 6*) there is given the angle $A = 39^\circ$, $B = 51^\circ$, and $AC = 266$.

To find the other leg, BC.

Radii compared $\left\{ \begin{array}{l} 39 \text{ --- } 61.97 \\ 51 \text{ --- } 65.62 \end{array} \right.$

1st proportion, $61.97 : 39 :: 65.62 : 41.297$
= artificial sine of 39°

2d proportion, $51 : 266 :: 41.3 : 215.4 =$
BC.

EXAMPLE III.

In the *oblique* triangle ABC, (*Fig. 2.*)

Let there be given $\left\{ \begin{array}{l} \text{Angle A } 30^\circ \\ \text{Angle B } 45^\circ \\ \text{Side BC } 290 \end{array} \right\}$ Required the
fides, AC
and AB.

If this problem were to be worked by the common rules (*Vid. § 1.*) it would be necessary to divide the triangle into two *right-angled triangles*, by means of a perpendicular (as CD) from C, on the side AB; then the natural radius must be found for the angle at B, by means of which, and the side CB, the length of the perpendicular is obtained:

tained : After this, with a second natural radius, find the side AC in the other right-angled triangle ; by subtracting the square of CD from that of AC, and extracting the square root, we determine the segment AD : and by a similar operation we learn the value of DB. Whence the length of the whole base is obtained.——Here, there is not only a very tedious course of calculation, * but the inconvenience of not being able to find the length of AB, *independently* of AC : which by the *new scheme*, is effected with the utmost ease ; as follows.

The angle at C = 105° ; the sine of this is the same as the sine of its supplement ; therefore substitute its supplement = 75° (for $180 - 105 = 75$).

Radii of A and C compared $\left\{ \begin{array}{l} 30 \text{——} 60 \\ 75 \text{——} 77.64 \end{array} \right.$

Then, 1) $60 : 30 :: 77.64 : 38.82$

2) $38.82 : 290 :: 75 : 560 = \text{length of AB.}$

* ——— *Longa est injuria, longæ Ambages.* ——— VIRG.

Again,

Again; to find the side AC, proceed thus.

Radii of A and B compared $\left\{ \begin{array}{l} 30 \text{ --- } 60 \\ 45 \text{ --- } 63.64 \end{array} \right.$

1) $60 : 30 :: 63.64 : 31.82$

2) $31.82 : 290 :: 45 : 410 = \text{side AC.}$

EXAMPLE IV.

In the oblique triangle ABC (*Fig. 3.*)

Given $\left\{ \begin{array}{l} A = 40^\circ, 15' \\ C = 35^\circ \\ AB = 380 \end{array} \right\}$ Required BC.

Radii compared $\left\{ \begin{array}{l} 40.25 \text{ --- } 62.29 \\ 35 \text{ --- } 61.02 \end{array} \right.$

1) $61.02 : 35 :: 62.29 : 35.7$

2) $35.7 : 380 :: 40.25 : 428 = \text{BC.}$

EXAMPLE V.

In the oblique triangle ABC (*Fig. 3.*)

Given $\left\{ \begin{array}{l} A = 27^\circ \\ B = 113 \\ BC = 98 \end{array} \right\}$ Required AC.

Note, $180 - 113 = 67 = \text{supplement of B.}$

Radii of A and B compared $\left\{ \begin{array}{l} 27 \text{ --- } 59.47 \\ 67 \text{ --- } 72.78 \end{array} \right.$

1) $59.47 : 27 :: 72.78 : 33$

2) $33 : 98 :: 67 : 198 = \text{AC.}$

From these examples, I flatter myself, it is evident, that the plan now proposed to the public, is of considerable service even in *oblique trigonometry*; not only, as it facilitates the problems which are referred to right-angled triangles, but as it affords a *direct* solution of one of its principal cases, *viz. two angles, and a side being given, to find the other two sides.*

I shall only add, that the usefulness of the table annexed is not confined to triangles; but may extend to those problems in general, which are resolved by finding the *sine*, *tangent* or *secant* of some given arch. For all these may be determined from hence, with sufficient accuracy, when the usual canon is not at hand. Suppose, for example, the given arch = 48° ; then its *sine* (with radius unity) must be the quotient of the arch divided by the tabular radius, $= \frac{48}{64.59} = .74314$; and its *cosine*, or the *sine* of 42° , $= \frac{42}{62.77}$: hence by known rules, we have the *verse-sine* $= 1 - \text{cosine}$; the *tangent* $= \frac{1 \times \text{sine}}{\text{cosine}}$; and the *secant* $= \frac{1}{\text{cosine}}$

T H E T A B L E.

A	R	D	A	R	D	A	R	D
1	57.3		31	60.19	.19	61	69.74	.46
2	57.31	.01	32	60.39	.2	62	70.22	.48
3	57.32	.01	33	60.59	.2	63	70.7	.48
4	57.34	.02	34	60.8	.21	64	71.2	.5
5	57.37	.03	35	61.02	.22	65	71.72	.52
6	57.4	.03	36	61.24	.22	66	72.24	.52
7	57.44	.04	37	61.48	.24	67	72.78	.54
8	57.48	.04	38	61.72	.24	68	73.34	.56
9	57.53	.05	39	61.97	.25	69	73.9	.56
10	57.59	.06	40	62.23	.26	70	74.49	.59
11	57.65	.06	41	62.49	.26	71	75.09	.6
12	57.72	.07	42	62.77	.28	72	75.7	.61
13	57.79	.07	43	63.05	.28	73	76.33	.63
14	57.87	.08	44	63.34	.29	74	76.98	.65
15	57.95	.08	45	63.64	.3	75	77.64	.66
16	58.04	.09	46	63.94	.3	76	78.32	.68
17	58.14	.1	47	64.26	.32	77	79.02	.70
18	58.25	.11	48	64.59	.33	78	79.74	.72
19	58.36	.11	49	64.92	.33	79	80.48	.74
20	58.47	.11	50	65.27	.35	80	81.23	.75
21	58.6	.13	51	65.62	.35	81	82.01	.78
22	58.73	.13	52	65.98	.36	82	82.8	.79
23	58.86	.13	53	66.36	.38	83	83.62	.82
24	59.	.14	54	66.74	.38	84	84.46	.84
25	59.15	.15	55	67.14	.4	85	85.32	.86
26	59.31	.16	56	67.54	.4	86	86.21	.89
27	59.47	.16	57	67.96	.42	87	87.12	.91
28	59.64	.17	58	68.39	.43	88	88.05	.93
29	59.81	.17	59	68.83	.44	89	89.01	.96
30	60.	.19	60	69.28	.45	90	90.	.99

MINUTES.

1	.016	31	.516
2	.033	32	.533
3	.05	33	.55
4	.066	34	.566
5	.083	35	.583
6	.1	36	.6
7	.116	37	.616
8	.133	38	.633
9	.15	39	.65
10	.166	40	.666
11	.183	41	.683
12	.2	42	.7
13	.216	43	.716
14	.233	44	.733
15	.25	45	.75
16	.266	46	.766
17	.283	47	.783
18	.3	48	.8
19	.316	49	.816
20	.333	50	.833
21	.35	51	.85
22	.366	52	.866
23	.383	53	.883
24	.4	54	.9
25	.416	55	.916
26	.433	56	.933
27	.45	57	.95
28	.466	58	.966
29	.483	59	.983
30	.5		



A P P E N D I X.

HAVING suggested in the beginning of this Essay, that from the analysis laid down in the second section, some *new formulae* might be raised for the solution of triangles, I shall here present the reader with *three* of the most useful equations, that have hitherto occurred to my inquiry.

It has been shewn (p. 11, and 13) that in any right-angled triangle, if a be the lesser angle in degrees, b = hypotenuse, b = longer leg, and c = less, then $a \times \frac{2b + b}{172c}$; and $b = b \times 1 - \frac{1.5aa}{10000}$,

From these equations, I have deduced the following, but shall conceal the process for the sake of exercising the learner: for convenience, put $x = 1 - \frac{b}{b}$, and $m = \frac{20000a}{10000 - 1.5aa}$

+ a ; then

$$a =$$

$$\left. \begin{aligned} a &= \sqrt{\frac{20000x}{3}} \\ b &= \frac{172c}{m} \\ \text{and } c &= \frac{bm}{172} \end{aligned} \right\} \text{ nearly.}$$

In applying these equations to practice, there is no great difficulty; the two last are most useful, because taken together they form a *compleat* case in trigonometry; whereas the first equation is limited, being of little value, when b and c , instead of b and b , are the given quantities. However, they all possess one convenient property (when rightly applied) that of discovering the thing sought, without subjecting us to the necessity of finding the *other parts* of a triangle. I shall give an illustration of each in numbers.

EQUATION 1st, $a = \sqrt{\frac{20000x}{3}}$.

Let there be a right-angled triangle, having $b = 520$, and $b = 432$; required the angles?

Here

Here $\frac{b}{b} = .83$; therefore, $x = .17$

$$\begin{array}{r}
 .17 \\
 \underline{20000} \\
 3) \underline{3400.00} \\
 1133.33 \quad (33.66 = a, \text{ or } 33^{\circ}, 40' \\
 \underline{9} \\
 63) \underline{233} \\
 189 \\
 666) \underline{4433} \quad 90 - a = 56^{\circ}, 20' \\
 3996 \\
 6726) \underline{43700} \\
 40356 \\
 3344
 \end{array}$$

EQUATION 2d. $b = \frac{172c}{m}$,

where m signifies $\frac{20000a}{10000 - 1.5aa} + a$.

Let there be given, in a right-angled triangle, $a = 37^{\circ}, 30'$, and $c = 85$; required b ?

(See above, p. 27.)

aa

[40]

$$\begin{array}{r} aa = 1406.25 \\ \frac{1}{2} = 703 \\ \hline 2109 \end{array} \qquad \begin{array}{r} 10000 \\ 2109 \\ \hline 7891 \end{array}$$

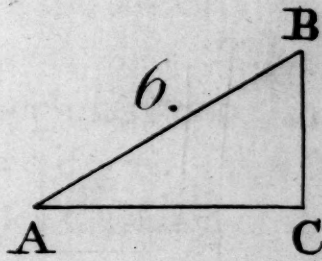
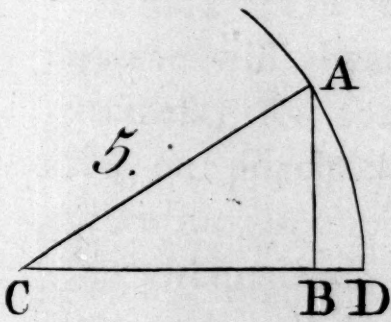
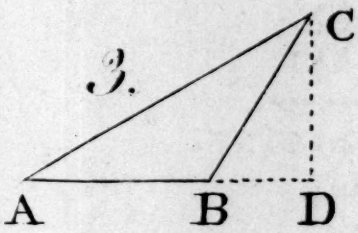
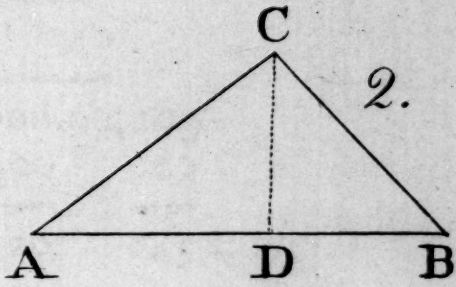
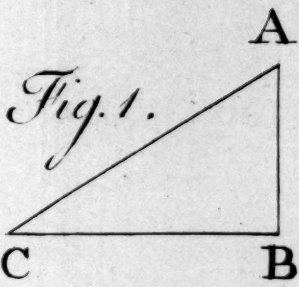
$$\begin{array}{r} 37.5 \\ 20000 \\ \hline 7891 \overline{) 750000.0} (95 \\ \underline{39810} \quad \underline{37.5} \\ 355 \quad 132.5 = m. \end{array}$$

$$\begin{array}{r} 172 \\ 85 = c \\ \hline 860 \\ 1376 \\ \hline 132.5 \overline{) 14620} (110.3 = b. \\ \underline{1370} \\ 4500 \\ \hline 625 \end{array}$$

EQUATION 3d. $c = \frac{bm}{17^2}$

In a right-angled triangle, suppose $a = 40^\circ$, and $b = 148$; required c ?

aa





$$\begin{array}{r} aa = 1600 \\ \frac{1}{2} = 800 \\ \hline 2400 \end{array} \qquad \begin{array}{r} 10000 \\ 2400 \\ \hline 7600 \end{array}$$

$$20000$$

$$40$$

$$76) 8000.00(105$$

$$400$$

$$40$$

$$20$$

$$145 = m$$

$$148 = b$$

$$145$$

$$740$$

$$592$$

$$148$$

$$172) 21460(124 = c$$

$$426$$

$$820$$

$$132$$

It must not be expected, that any of these *formulae* will always give the answer with exactness; for they depend, like *Wilson's* rule, on principles of approximation, and approximation implies some degree of defect. This circumstance increases the value of the *new table*, which insures an accuracy unattainable by the common schemes.

F I N I S.